VARIATIONAL PROBLEMS IN THE OPTIMIZATION OF CONTROL PROCESSES IN SYSTEMS WITH BOUNDED COORDINATES

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In the design of optimal control processes the restrictions considered are in most cases only those imposed on the control parameters. It is only in a few studies, in the solution of particular problems, that attempts have been made to consider restrictions on the coordinates. In the general formulation, the problem of optimization when there are restrictions on the coordinates has been studied in [1] and [2].

In the present article we discuss the variational formulation of problems in the optimization of control processes in systems whose coordinates and parameters may be bounded. The article lists the fundamental [2] necessary conditions for a minimum of the appropriate functionals making it possible to construct solutions of such problems.

1. Introduction. The general optimization problem for control processes is usually formulated for systems described by the differential equations [2]

$$\dot{x}_s = f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (s = 1, \ldots, n) \qquad (1.1)$$

which will be supplemented by the finite relationships [3,4]

$$\psi_k = \psi_k(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r) \qquad (1.2)$$

Here $x_s(t)$ are the coordinates of the system and $u_k(t)$ are the control parameters. The derivatives of the latter do not enter into the problem equations.

Optimization problems for systems with bounded control parameters in variational formulation have been studied in [3,4]. These articles describe methods for the transition to open regions of variation of the control parameters and show that in optimal operation the control parameters u_k may assume values corresponding to boundary points of the closed region U^* of admissible values. The necessity of considering boundary values of the control parameters in these problems did not complicate their solution very much, since the systems under investigation are described by the same equations both inside and outside the boundary of the region U^*.

A similar result is obtained in optimization problems involving control processes in systems with bounded coordinates: the coordinates and the control parameters of such systems, in optimal operation, may also assume values corresponding to the boundaries of the closed regions X^* and U^* of permissible variation of coordinates and control parameters. In this case, however, the fact that it is possible for the coordinates to go beyond the boundary may greatly complicate the problem of designing optimal modes of operation. The reasons for this are the following:

The behavior of the integral curves of a system with bounded coordinates and given control parameters is defined by Equations (1.1). It may happen that these curves do not include any which lie, even in part, on the boundary of the closed region of permissible variation of the coordinates, since Equations (1.1) and the equation of this boundary may not be valid simultaneously. Furthermore, Equations (1.1) may change their form or their order when the representative point goes beyond the boundary.

For this reason, before studying the construction of relationships defining optimal modes of operation, we should pause to clarify the nature of the restrictions on the coordinates with which we shall have to deal in the solution of optimization problems.

For the sake of definiteness we assume that the region X^* of permissible variation of the coordinates $x_1, \ldots, x_n$ is defined by the inequality [1,2]

$$\vartheta(x_1, \ldots . x_n) \leqslant 0 \tag{1.3}$$

It may happen that this inequality does not reflect the internal properties of this system, and its coordinates in arbitrary modes of motion may go beyond the limits of the region X^*. We may then say that the requirement (1.3) is imposed externally on the system.

We may obtain an idea of these restrictions from a consideration of the following example. Let an optimal process be constructed in a system without consideration of restrictions on the coordinates, and let the coordinates in this process assume undesirable or prohibited values. Then the optimization problem may be restated, and this new formulation must be such as to reflect the fact that certain definite values of the coordinates are prohibited. In some cases this will lead to inequality of the form (1.3).

Restrictions on the coordinates imposed externally upon the system will hereafter be called restrictions of the first type.

Restrictions of the second type upon the coordinates reflect the presence in the system of restrictive factors, such as stops, saturation zones, etc. In this case, in any motion of the system its coordinates cannot go beyond the limits of a closed region of permissible coordinate variation.

We must deal with restrictions of the second type on the coordinates, for example, when an indirect system for controlling an aircraft contains stops on the control surfaces. Then the control surfaces cannot assume positions outside the intervals defined by the stops.

Similar restrictions are found in a control system for the pressure in the boiler when safety valves are used, and in many other cases.

It should be noted that restrictions of both types may be represented by identical inequalities. This may lead to the erroneous conclusion that the mathematical description of these restrictions in optimization problems will be identical. In actual fact, of course, this is not true.

For example, we may try using restrictions of the first type on the control surface coordinates to take account of the presence of stops in the control system for the motion of an aircraft. But such restrictions will keep the coordinate values at the boundary independently of the stops, and these stops may be removed. In such a formulation the control surfaces do not exert any pressure on the stops. This pressure may be taken into account by means of restrictions of the second type.

References [1,2] study in detail the coordinate limitations imposed externally (first type), represented by the inequality (1.3). It is established in these works that such a restriction may be realized if the optimal trajectory is to consist of a finite number of segments located within the region X^* or on its boundary. It is shown that in order to have a segment of the trajectory in the interval $t_1 \leqslant t \leqslant t_2$ lie on the boundary it is necessary and sufficient to require that the equality

$$\vartheta\,[x_1(t_1), \ldots, x_n(t_1)] = 0 \tag{1.4}$$

be satisfied at time $t = t_1$ and that the relations

$$\psi_{r+1} = \psi_{r+1}(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = \sum_{s=1}^{n} \frac{\partial \vartheta}{\partial x_s} f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \tag{1.5}$$

be satisfied over the entire interval $t_1 \leqslant t \leqslant t_2$.

This function ψ_{r+1} gives the projection of the phase velocity of the system on the external normal to the boundary $\vartheta = 0$ of the region (1.3). Since this projection is equal to zero and the phase velocity is tangent to the boundary, the representative point will not exert a "pressure" on the boundary.

The equality (1.5) imposes on the control parameters an additional relation of the form (1.2). Therefore, the end $t = t_2$ of the interval $t_1 \leqslant t \leqslant t_2$ coincides with the instant immediately to the right of which, at $t = t_2 + 0$, there is no point at which Equation (1.5) can be satisfied. The values of the coordinates $x_s(t_2)$ at the point $t = t_2$ are related by the equation

$$\psi_{r+1}[x_1(t_2), \ldots, x_n(t_2), u_1(t_2), \ldots, u_m(t_2), t_2] = 0 \tag{1.6}$$

If the representative point touches the boundary of the region X^*, which corresponds to a restriction of the second type, it will remain on the boundary until its "pressure" on the boundary changes sign. Therefore, in the case of restrictions of the second type, the representative point will always move along the boundary so long as the normal component of the phase velocity is non-negative:

$$\psi_{r+1}(x_1, \ldots, x_n u_1, \ldots, u_m, t) \geqslant 0 \tag{1.7}$$

At the boundary points the equality $\vartheta = 0$ is satisfied.

At time $t = t_2$ the phase velocity touches the boundary, so that at $t = t_2$ Formula (1.6) is again valid, with $\psi_{r+1}(t_2 + 0) < 0$.

Since in the case of restrictions of the second type the representative point may produce a pressure on the boundary of the region X^*, the motion of the system along this boundary can be described by equations different from the equations of motion of the point within the region X^*. In the simplest problem of this type, which will be considered hereafter, these equations are constructed by means of the equations of motion of the system within the region X^*. However, in the general case this is not true and setting up the equations of motion of the system for boundary points requires additional investigation in each concrete case.

We shall hereafter study systems with restrictions of the second type, given by the inequalities

$$x_{s'} \leqslant 0 \tag{1.8}$$

where it will be assumed that when the representative point passes from the interior to the boundary, the motion of the system is described by Equations (1.1), in which the equation numbered s' is replaced by the equality $x_{s'} = 0$.

By this method we can investigate restrictions described by the inequalities

$$X_{s'}^{(1)} \leqslant x_{s'}(t) \leqslant X_{s'}^{(2)} \tag{1.9}$$

2. Statement of the problem. In solving optimization problems for control processes with bounded coordinates we shall use the following general variational formulation:

Among the coordinates $x_1, \ldots, x_n$ which do not go beyond the limits of some closed region X^* of allowable values and the control functions $u_1, \ldots, u_m$, satisfying in the interval $t_0 \leqslant t \leqslant T$ the system of equations

$$g_s = \dot{x}_s - f_s(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (s = 1, \ldots, n) \tag{2.1}$$

and the finite relations

$$\psi_k = \psi_k(x_1, \ldots, x_n, u_1, \ldots, u_m, t) = 0 \qquad (k = 1, \ldots, r) \tag{2.2}$$

and related by the equations

$$\varphi_l = \varphi_l[x_1(t_0), \ldots, x_n(t_0), t_0, x_1(T), \ldots, x_n(T), T] = 0 \tag{2.3}$$
$$(l = 1, \ldots, p \leqslant 2n + 1)$$

we must find those which will give the functional

$$J = g[x_1(t_0), \ldots, x_n(t_0) t_0, x_1(T), \ldots, x_n(T), T] + \int_{t_0}^{T} f_0(x_1, \ldots, x_n, u_1, \ldots, u_m, t)\, dt \tag{2.4}$$

a minimum value.

In this formulation no special emphasis was placed on the possibility that the control u_k may lie in a given closed region of permissible variation. The reason for this is that the methods described in [3-7] make it possible, by constructing auxiliary equalities of the form (2.2), to pass to an open region for the control parameters. Hereafter we shall consider this passage to have been completed.

We may try by analogous methods to pass to an open region of coordinate variation. However, such an approach will merely enable us to establish that the system coordinates in optimal modes of operation may assume values within and on the boundary of the region X^*, and to obtain equations which define segments of optimal trajectories located within this region. The equations for segments lying on the boundary cannot be constructed by this method. This is not surprising, since when the coordinates of the system pass from the interior to the boundary, as has already been shown, the equations of motion of the system may change, and this is not taken into account in such a construction. The above statement of the problem must therefore be modified.

We shall consider an optimal trajectory to consist of a finite number of segments located either in the interior or on the boundary of the region X^*. For the sake of definiteness, all the functions entering into Equations (2.1) and (2.2) which correspond to the boundary of the region X^* will hereafter be written with a superscript zero. Functions associated with a segment to the left of the boundary segment will be marked with a minus sign, and functions corresponding to a segment to the right of the boundary segment will be marked with a plus sign.

If we are dealing with a restriction of the first type (an external restriction), for coordinate values on the boundary of the region X^*, Equations (2.2) must be supplemented by the equality (1.5). In addition, we must take into consideration the condition (1.4), which defines the time at which the coordinates pass from the interior to the boundary, and the equality (1.6), which specifies the time at which the coordinates pass from the boundary into the interior of the region X^*.

For restrictions of the second type the problem becomes more complicated, since when the coordinates of the system pass from the interior to the boundary, its equations of motion may change. In the simplified case which we discussed above, the equations

$$g_s^0 = \dot{x}_s^0 - f_s^0(x_1^0, \ldots, x_n^0, u_1^0, \ldots, u_m^0, t) = 0 \qquad (s \neq s'), \quad g_{s'} = x_{s'} = 0 \qquad (2.5)$$

will hold on the boundary.

The instant $t = t_1$ when the system coordinates pass from the interior to the boundary is defined by the equality

$$x_{s'}(t_1) = 0 \qquad (2.6)$$

and at the instant $t = t_2$ when they pass from the boundary to the interior of the region, we will have

$$f_{s'}\,[x_1(t_2), \ldots, x_n(t_2), u_1(t_2), \ldots, u_m(t_2), t_2] = 0 \tag{2.7}$$

and $f_{s'}$ will change sign at this point. Inequalities of a different form may be considered in a similar manner.

Equations (2.1) and (2.2), for suitable choice of the numbers n and r, describe the behavior of the system in either of the above cases.

In this modified formulation the optimization problem for control processes in systems with bounded coordinates becomes a problem of the Mayer-Bolza type in the calculus of variations [8]. By comparison with the cases considered earlier [3, 4] it is considerably complicated by the necessity of taking into consideration the difference between the equations of motion on different segments of the integral curves. In this respect it is reminiscent of the optimization problem for control processes in the case of equations with discontinuous right-hand sides [9].

In the book by Bliss [8] and in [4] and [9], descriptions are given for the process of establishing the necessary conditions in variational problems in optimization. The corresponding reasoning and calculations may, of course, be extended to the cases considered here. However, even a brief discussion of these would occupy a great deal of space and is not given in the present article, although the results of such an extension, with the appropriate explanations, are used in the rest of the article.

As was done earlier, we shall consider the necessary condition for stationary state of the functional J and Weierstrass's necessary condition for its strong minimum. It will be assumed, of course, that all the requirements usually imposed in the calculus of variation on the functions entering into the formulation of the problem are satisfied. The functions which will make the functional J a minimum will be sought among the continuous coordinates $x_s(t)$ with piecewise continuous derivatives $\dot{x}_s(t)$ and among piecewise continuous control parameters $u_k(t)$.

3. Necessary condition for stationary state of the functional J. Restrictions of the first type on the coordinates. For the sake of simplicity it will first be assumed that in the interval $t_0 \leqslant t \leqslant T$ there is only one point $t = t_1$ at which the coordinates of the system pass from the interior of the region X^* to its boundary and that there are no other corner points [4]. For definiteness, we shall consider the closed region X^* to be defined by the inequality (1.3). This will enable us to make a comparison between the formulas obtained below and the relations with the corresponding results established in

[1] and [2].

In the subinterval $t_0 \leqslant t \leqslant t_1$ the equations

$$g_s^- = \dot{x}_s^- - f_s^-(x_1^-, \ldots, x_n^-, u_1^-, \ldots, u_m^-, t) = 0 \qquad (s = 1, \ldots, n) \qquad (3.1)$$

$$\psi_k^- = \psi_k^-(x_1^-, \ldots, x_n^-, u_1^-, \ldots, u_m^-, t) = 0 \qquad (k = 1, \ldots, r) \qquad (3.2)$$

will hold, and in the subinterval $t_1 \leqslant t \leqslant T$ they must be replaced by the system

$$g_s^\circ = \dot{x}_s^\circ - f_s^\circ(x_1^\circ, \ldots, x_n^\circ, u_1^\circ, \ldots, u_m^\circ, t) = 0 \qquad (s = 1, \ldots, n) \qquad (3.3)$$

$$\psi_k^\circ = \psi_k(x_1^\circ, \ldots, x_n^\circ, u_1^\circ, \ldots, u_m^\circ, t) = 0 \qquad (k = 1, \ldots, r+1) \qquad (3.4)$$

in which ψ_{r+1}° is defined by the relation (1.5).

It should be particularly emphasized that for any $r < m$ we may have cases in which Equations (3.4) will have solutions with respect to the control parameters $u_k(t)$ in the region of permissible values. For $r = m - 1$ the problem of optimizing the system motion corresponding to the boundary of the region X^* may also become meaningless because these equalities may be satisfied only by one unique system of permissible functions $u_k(t)$. Hereafter, therefore, we shall consider Equations (3.4) to be satisfied by permissible control parameters $u_k(t)$ in a non-unique manner.

At times $t = t_1$ Equation (1.4) must be satisfied, and the functional I used in constructing the necessary condition for stationary state of the functional J must be taken in the following form [9]

$$I = \varphi + \nu_1 \vartheta [x_1(t_1), \ldots, x_n(t_1)] + \int_{t_0}^{t_1} L^- dt + \int_{t_1}^{T} L^\circ dt \qquad (3.5)$$

In Formula (3.5) the following notation is used

$$\varphi = g + \sum_{l=1}^{p} \rho_l \varphi_l \qquad (3.6)$$

$$L^- = f_0^- + \sum_{s=1}^{n} \lambda_s^- g_s^- - \sum_{k=1}^{r} \mu_k^- \psi_k^- = \sum_{s=1}^{n} \lambda_s^- \dot{x}_s^- - H^- \qquad (3.7)$$

$$L^\circ = f_0^\circ + \sum_{s=1}^{n} \lambda_s^\circ g_s^\circ - \sum_{k=1}^{r+1} \mu_k^\circ \psi_k^\circ = \sum_{s=1}^{n} \lambda_s^\circ \dot{x}_s^\circ - H^\circ - \mu_{r+1}^\circ \psi_{r+1}^\circ \qquad (3.8)$$

$$H^- = H_\lambda^- + H_\mu^- = \sum_{s=0}^{n} \lambda_s^- f_s^- + \sum_{k=1}^{r} \mu_k^- \psi_k^- \qquad (\lambda_0^- = -1) \qquad (3.9)$$

$$H^{\circ} = H_{\lambda}^{\circ} + H_{\mu}^{\circ} = \sum_{s=0}^{n} \lambda_s^{\circ} f_s^{\circ} + \sum_{k=1}^{r} \mu_k^{\circ} \psi_k^{\circ} \qquad (\lambda_0^{\circ} = -1) \tag{3.10}$$

and $\lambda_s^{-}(t)$, $\lambda_s^{\circ}(t)$, $\mu_k^{-}(t)$, $\mu_k^{\circ}(t)$, ρ_l and ν_1 are undetermined Lagrangean multipliers suitable for calculation.

The stationary state condition is obtained by equating the first variation of the functional I to zero and is represented by the equality $\Delta I = 0$.

Substituting into Equation (3.5) the functions L^{-} and L° from Formulas (3.7) and (3.8) and constructing this variation, we arrive at the equation

$$\Delta I = \Delta\varphi + \nu_1 \Delta\vartheta + (f_0^{-} - f_0^{\circ})_{t_1} \delta t_1 - (f_0)_{t_0} \delta t_0 + (f_0)_T \delta T +$$
$$+ \int_{t_0}^{t_1} \Big[\sum_{s=1}^{n} \lambda_s^{-} \delta \dot{x}_s^{-} - \delta H^{-}\Big] dt + \int_{t_1}^{T} \Big[\sum_{s=1}^{n} \lambda_s^{\circ} \delta \dot{x}_s^{\circ} - \delta H^{\circ} - \mu_{r+1}^{\circ} \delta \psi_{r+1}^{\circ}\Big] dt$$

so that after the sums under the integral signs are integrated by parts once and the variations of the individual functions are expanded, we shall have

$$\Delta I = \sum_{s=1}^{n} \Big[\frac{\partial \varphi}{\partial x_s(t_0)} - \lambda_s(t_0)\Big] \Delta x_s(t_0) + \sum_{s=1}^{n} \Big[\frac{\partial \varphi}{\partial x_s(T)} + \lambda_s(T)\Big] \Delta x_s(T) +$$
$$+ \Big[\frac{\partial \varphi}{\partial t_0} + (H_\lambda)_{t_0}\Big] \delta t_0 + \Big[\frac{\partial \varphi}{\partial T} - (H_\lambda)_T\Big] \delta T +$$
$$+ \sum_{s=1}^{n} \Big[\lambda_s^{-}(t_1) - \lambda_s^{\circ}(t_1) + \nu_1 \frac{\partial \vartheta}{\partial x_s(t_1)}\Big] \Delta x_s(t_1) + [H_\lambda^{\circ} - H_\lambda^{-}]_{t_1} \delta t_1 -$$
$$- \int_{t_0}^{t_1} \Big[\sum_{s=1}^{n} \Big(\dot{\lambda}_s^{-} + \frac{\partial H^{-}}{\partial x_s^{-}}\Big) \delta x_s^{-} - \sum_{k=1}^{m} \frac{\partial H^{-}}{\partial u_k^{-}} \delta u_k^{-}\Big] dt -$$
$$- \int_{t_1}^{T} \Big[\sum_{s=1}^{n} \Big(\dot{\lambda}_s^{\circ} + \frac{\partial H^{\circ}}{\partial x_s^{\circ}} + \frac{\partial \psi_{r+1}^{\circ}}{\partial x_s} \mu_{r+1}^{\circ}\Big) \delta x_s^{\circ} + \sum_{k=1}^{m} \Big(\frac{\partial H^{\circ}}{\partial u_k^{\circ}} + \mu_{r+1}^{\circ} \frac{\partial \dot{\psi}_{r+1}}{\partial u_k^{\circ}}\Big) \delta u_k^{\circ}\Big] dt \tag{3.11}$$

Here we use Equation (3.1) and (3.3) and the notation of (3.9) and (3.10); where no confusion results, we omit indices and use, for example, $(f_0^{-})_{t_1}$ to represent the value of function f_0^{-} at the point $t = t_1$.

The variation (3.11) must be taken equal to zero, and therefore the coefficients for all independent variations of the variables will be equal to zero. The corresponding coefficients for dependent variations of the variables may be made zero by the choice of Lagrangean multipliers. After these operations we obtain the system of equations

$$\dot{\lambda}_s^- + \frac{\partial H^-}{\partial x_s^-} = 0, \qquad \dot{\lambda}_s^\circ + \frac{\partial H^\circ}{\partial x_s^\circ} + \mu_{r+1}^\circ \frac{\partial \psi_{r+1}^\circ}{\partial x_s^\circ} = 0 \qquad (s = 1, \ldots, n) \tag{3.12}$$

with the conditions

$$\frac{\partial H^-}{\partial u_k^-} = 0, \qquad \frac{\partial H^\circ}{\partial u_k^\circ} + \mu_{r+1}^\circ \frac{\partial \psi_{r+1}^\circ}{\partial u_k^\circ} = 0 \qquad (k = 1, \ldots, m) \tag{3.13}$$

the equalities

$$\frac{\partial \varphi}{\partial x_s(t_0)} - \lambda_s(t_0) = 0, \qquad \frac{\partial \varphi}{\partial x_s(T)} + \lambda_s(T) = 0 \qquad (s = 1, \ldots, n) \tag{3.14}$$

$$\frac{\partial \varphi}{\partial t_0} + (H_\lambda)_{t_0} = 0, \qquad \frac{\partial \varphi}{\partial T} - (H_\lambda)_T = 0 \tag{3.15}$$

and the Erdmann-Weierstrass conditions

$$\lambda_s^-(t_1) - \lambda_s^\circ(t_1) + \nu_1 \frac{\partial \vartheta}{\partial x_s(t_1)} = 0 \qquad (s = 1, \ldots, n) \tag{3.16}$$

$$(H_\lambda^-)_{t_1} - (H_\lambda^\circ)_{t_1} = 0 \tag{3.17}$$

In solving optimization problems we must also make use of Equations (3.1) to (3.4), the relation (2.3), the equality (1.4), and the continuity conditions on the coordinates

$$x_s^-(t_1) = x_s^\circ(t_1) \qquad (s = 1, \ldots, n) \tag{3.18}$$

Then the number $4n + 2m + 2r + 1$ of Equations (3.1) to (3.4), (3.12), (3.13) will be equal to the number of functions $x_s^-(t)$, $x_s^\circ(t)$, $u_k^-(t)$, $u_k^\circ(t)$, $\lambda_s^-(t)$, $\lambda_s^\circ(t)$, $\mu_k^-(t)$, $\mu_k^\circ(t)$.

The solution of the differential Equations (3.1), (3.3) and (3.12) contains $4n$ constants of integration; to find these, in addition to the p multipliers ρ_e and the values of t_0, t_1, T, we must use the $4n + p + 3$ conditions (3.14), (3.16), (3.18), (2.3), (3.15) and (3.17).

Let us now assume that in the interval $t_0 \leqslant t \leqslant T$ there is only one point $t = t_2$ at which the coordinates of the system pass from the boundary to the interior of the region X^*, and there are no other corner points.

Then in the subinterval $t_0 \leqslant t \leqslant t_2$ Equations (3.3) and (3.4) are valid, and in the subinterval $t_2 \leqslant t \leqslant T$ we will have

$$g_s^+ = \dot{x}_s^+ - f_s^+(x_1^+, \ldots, x_n^+,\ u_1^+, \ldots, u_m^+, t) = 0 \qquad (s = 1, \ldots, n) \tag{3.19}$$

$$\psi_k^+ = \psi_k^+(x_1^+, \ldots, x_n^+, \quad u_1^+, \ldots, \quad u_m^+, \quad t) = 0 \qquad (k = 1, \ldots, r) \tag{3.20}$$

with the relation (1.5) satisfied at time $t = t_2$.

In this case the functional I, which is used in the construction of the stationary state condition, should be taken in the form

$$I = \varphi + \int_{t_0}^{t_2} L^\circ\, dt + \int_{t_2}^{T} L^+\, dt \tag{3.21}$$

(since the point $t = t_2$ will not be a corner point). Here φ is defined by the equality (3.6), and the function L° by the relation (3.8), in which H° is expressible in the form (3.10).

For L^+ we obtain the formula

$$L^+ = f_0^+ + \sum_{s=1}^{n} \lambda_s^+ g_s^+ + \sum_{k=1}^{r} \mu_k^+ \psi_k^+ = \sum_{s=1}^{n} \lambda_s^+ \dot{x}_s^+ - H^+ \tag{3.22}$$

where

$$H^+ = H_\lambda^+ + H_\mu^+ = \sum_{s=1}^{n} \lambda_s^+ f_s^+ + \sum_{k=1}^{r} \mu_k^+ \psi_k^+ \qquad (\lambda_0^+ = -1) \tag{3.23}$$

and $\lambda_s^+(t)$, $\mu_k^+(t)$ are undetermined Lagrangean multipliers.

Formulating and equating to zero the first variation ΔI of the functional I and repeating all of the above operations, we obtain the equations

$$\dot{\lambda}_s^+ + \frac{\partial H^+}{\partial x_s^+} = 0, \qquad \dot{\lambda}_s^\circ + \frac{\partial H^\circ}{\partial x_s^\circ} + \mu_{r+1}^\circ \frac{\partial \psi_{r+1}^\circ}{\partial x_s^\circ} = 0 \qquad (s = 1, \ldots, n) \tag{3.24}$$

$$\frac{\partial H^+}{\partial u_k^+} = 0, \qquad \frac{\partial H^\circ}{\partial u_k^\circ} + \mu_{r+1}^\circ \frac{\partial \psi_{r+1}^\circ}{\partial u_k^\circ} = 0 \qquad (k = 1, \ldots, m) \tag{3.25}$$

Equations (3.15) and (3.16), and the Erdmann-Weierstrass conditions

$$\lambda_s^\circ(t_2) - \lambda_s^+(t_2) = 0 \quad (s = 1, \ldots, n), \qquad (H_\lambda^\circ - H_\lambda^+)_{t_2} = 0 \tag{3.26}$$

We count the number of equations and functions and the number of constants and conditions which determine them, as in the previous case.

We might have assumed at the very beginning that the interval

$t_0 \leqslant t \leqslant T$ contains two points $t = t_1$ and $t = t_2$ of the type described above. This would have complicated the calculation but would not have changed the final result. Even more general assumptions do not change the results.

The discontinuities of the control parameters $u_k(t)$, if any exist in the interval $t_0 \leqslant t \leqslant T$, are studied in the same way as in the articles [4,9]. The special case in which the instant at which there is a discontinuity in the control parameter coincides with the instant at which the system coordinates pass to or from the boundary of the region X^* is investigated in the same way as in the article [9].

4. Stationary state condition for the functional J. Restrictions of the second type on the coordinates. Let us consider restrictions which are simpler but are more frequently encountered; these are defined by the inequalities (1.8). For the sake of simplicity we shall first assume that there is only one restriction $x_1 \leqslant 0$, and we shall consider $s' = 1$, which of course can always be done by means of a change in the numbering of the variables.

We shall again assume that the interval $t_0 \leqslant t \leqslant T$ contains only one point $t = t_1$ at which the coordinates of the system pass from the interior to the boundary of the region X^*. Then in the subinterval $t_0 \leqslant t \leqslant t_1$ we must use Equations (3.1) and (3.2), and in the interval $t_1 \leqslant t \leqslant T$ we shall have

$$g_1^\circ = x_1^\circ = 0 \tag{4.1}$$

$$g_s^\circ = \dot{x}_s^\circ - f_s^\circ(x_1^\circ, \ldots, x_n^\circ, u_1^\circ, \ldots, u_m^\circ, t) = 0 \qquad (s = 2, \ldots, n) \tag{4.2}$$

$$\psi_k^\circ = \psi_k^\circ(x_1^\circ, \ldots, x_n^\circ, u_1^\circ, \ldots, u_m^\circ, t) = 0 \qquad (k = 1, \ldots, r) \tag{4.3}$$

in the functions f_s° and ψ_k° we must set $x_1^\circ = 0$. At time $t = t_1$ the equality $x_1(t_1) = 0$ is satisfied. The functional I will have the form

$$I = \varphi + \nu_1 x_1(t_1) + \int_{t_0}^{t_1} L^- \, dt + \int_{t_1}^{T} L^\circ \, dt \tag{4.4}$$

where φ is defined by the equality (3.6), L^- and H^- are defined by Formulas (3.7) and (3.9), and the function L° is equal to

$$L^\circ = f_0^\circ - \lambda_1^\circ x_1^\circ + \sum_{s=1}^{n} \lambda_s^\circ g_s^\circ - \sum_{k=1}^{r} \mu_k^\circ \psi_k^\circ = \sum_{s=2}^{n} \lambda_s^\circ \dot{x}_s^\circ - H^\circ \tag{4.5}$$

where

$$H^\circ = H_\lambda^\circ + H_\mu^\circ = \sum_{s=0}^{n} \lambda_s^\circ f_s^\circ + \sum_{k=1}^{r} \mu_k^\circ \psi_k^\circ \qquad (f_1^\circ = x_1^\circ = 0, \ \lambda_0^\circ = -1) \tag{4.6}$$

The first variation ΔI of the functional I is represented by the equality

$$\Delta I = \Delta\varphi + \nu_1 \Delta x_1(t_1) + (f_0^-)_{t_1}\delta t_1 - (f_0)_{t_0}\delta t_0 + (f_0)_T \delta T - (f_0^\circ)_{t_1}\delta t_1 + \\ + \int_{t_0}^{t_1} \Big(\sum_{s=1}^{n} \lambda_s^- \delta \dot{x}_s^- - \delta H^-\Big) dt + \int_{t_1}^{T} \Big(\sum_{s=2}^{n} \lambda_s^\circ \delta \dot{x}_s^\circ - \delta H^\circ\Big) dt$$

so that after transformations similar to the above, we obtain

$$\Delta I = \sum_{s=1}^{n} \Big[\frac{\partial\varphi}{\partial x_s(t_0)} - \lambda_s(t_0)\Big] \Delta x_s(t_0) + \sum_{s=1}^{n} \Big[\frac{\partial\varphi}{\partial x_s(T)} + \lambda_s(T)\Big] \Delta x_s(T) + \\ + \sum_{s=2}^{n} [\lambda_s^-(t_1) - \lambda_s^\circ(t_1)] \Delta x_s(t_1) + [\lambda_1(t_1) + \nu_1] \Delta x_1(t_1) + \Big[\frac{\partial\varphi}{\partial t_0} + (H_\lambda)_{t_0}\Big]\delta t_0 + \\ + \Big[\frac{\partial\varphi}{\partial T} - (H_\lambda)_T\Big]\delta T + [H_\lambda^\circ - H_\lambda^-]_{t_1}\delta t_1 - \int_{t_0}^{t_1} \Big[\sum_{s=1}^{n}\Big(\dot{\lambda}_s^- + \frac{\partial H^-}{\partial x_s^-}\Big)\delta x_s^- + \\ + \sum_{k=1}^{m} \frac{\partial H^-}{\partial u_k^-}\delta u_k^-\Big] dt - \int_{t_1}^{T} \Big[\sum_{s=2}^{n}\Big(\dot{\lambda}_s^\circ + \frac{\partial H^\circ}{\partial x_s^\circ}\Big)\delta x_s^\circ + \sum_{k=1}^{m} \frac{\partial H^\circ}{\partial u_k^\circ}\delta u_k^\circ\Big] dt \qquad (4.7)$$

Again equating to zero the coefficients of the independent variations and choosing the Lagrangean multipliers in such a way as to make the coefficients of the dependent variations of the variables vanish, we find the equations

$$\dot{\lambda}_s^- + \frac{\partial H^-}{\partial x_s^-} = 0, \quad (s = 1, \ldots, n), \qquad \dot{\lambda}_s^\circ + \frac{\partial H^\circ}{\partial x_s^\circ} = 0 \quad (s = 2, \ldots, n) \qquad (4.8)$$

$$\frac{\partial H^-}{\partial u_k^-} = 0, \qquad \frac{\partial H^\circ}{\partial u_k^\circ} = 0 \qquad (k = 1, \ldots, m) \qquad (4.9)$$

the end conditions

$$\frac{\partial\varphi}{\partial x_s(t_0)} - \lambda_s(t_0) = 0, \qquad \frac{\partial\varphi}{\partial x_s(T)} + \lambda_s(T) = 0 \qquad (s = 1, \ldots, n) \qquad (4.10)$$

$$\frac{\partial\varphi}{\partial t_0} + (H_\lambda)_{t_0} = 0, \qquad \frac{\partial\varphi}{\partial T} - (H_\lambda)_T = 0 \qquad (4.11)$$

and the Erdmann-Weierstrass conditions

$$\lambda_s^-(t_1) = \lambda_s^\circ(t_1), \quad (s = 2, \ldots, n), \qquad \lambda_1^-(t_1) + \nu_1 = 0 \qquad (4.12)$$

$$(H_\lambda^- - H_\lambda^\circ)_{t_1} = 0 \qquad (4.13)$$

In order to solve the problem we must add to the above the equations (3.1), (3.2), (4.1), (4.2), (4.3), and the conditions (2.3) and (3.19) for $s \neq 1$, and the equalities $x_s(t_1) = 0$.

We consider in a similar manner the case in which the interval $t_0 \leqslant t \leqslant T$ contains one instant $t = t_2$ at which the coordinates pass from the boundary to the interior of the region X^*. In this case the point $t = t_2$ must satisfy the condition

$$f_1[x_1(t_2), \ldots, x_n(t_2), u_1(t_2), \ldots, u_m(t_2), t_2] = 0 \tag{4.14}$$

Therefore, after appropriate calculations, we obtain the equations

$$\dot{\lambda}_s{}^\circ + \frac{\partial H^\circ}{\partial x_s{}^\circ} = 0 \quad (s = 2, \ldots, n), \qquad \dot{\lambda}_s{}^+ + \frac{\partial H^+}{\partial x_s{}^+} = 0 \quad (s = 1, \ldots, n) \tag{4.15}$$

$$\frac{\partial H^\circ}{\partial u_k{}^\circ} = 0, \qquad \frac{\partial H^+}{\partial u_k{}^+} = 0 \qquad (k = 1, \ldots, m) \tag{4.16}$$

The end conditions will retain their form (4.10) and (4.11), and the Erdmann-Weierstrass conditions will be expressible in the form

$$\lambda_s{}^\circ(t_2) - \lambda_s{}^+(t_2) = 0 \qquad (s = 2, \ldots, n), \qquad -\lambda_1{}^+(t_2) = 0 \tag{4.17}$$

$$(H_\lambda{}^\circ - H_\lambda{}^+)_{t_2} = 0 \tag{4.18}$$

It should be noted once more that assumptions of a more general type than those which were made above will not change the results but may considerably complicate the process of obtaining them. The points of discontinuity of the control parameters are studied in the same way as in [1]. The instants of discontinuity of the control parameters $u_k(t)$ and the instants at which the representative point passes to the boundary or to the interior of the region X^* are considered in a manner similar to the discussion of [9]. In this case the corresponding relations will be identical with those listed in this section.

We use similar methods to construct the expanded form of the stationary state condition if the region X^* is defined by a number of inequalities of the form (1.8). Here we must consider the various portions of the boundary of the region X^* defined by various equalities of the type $x_{s'} = 0$, corresponding to the above indicated inequalities, as well as the cases in which several of these equalities defining the boundary are satisfied simultaneously.

A transition to closed regions X^* defined by the inequalities (1.9) will likewise produce no significant complication. In this case $f_{s'} \geqslant 0$ for $x_{s'} = X_{s'}{}^{(2)}$ and $f_{s'} \leqslant 0$ for $x_{s'} = X_{s'}{}^{(1)}$. All the other results remain valid.

5. Weierstrass's necessary condition for a strict minimum of the functional J. If we repeat the reasoning and calculations described in Bliss's book [8] and the discussion [4] as applied to optimization problems with restrictions on the control parameters, then in the present case of restricted coordinates Weierstrass's necessary condition for a strict minimum of the functional J can be represented by the inequality

$$E \geqslant 0 \tag{5.1}$$

in which E is the Weierstrass function, defined by means of the formula

$$\begin{aligned} E = L(x_1, \ldots, x_n, \dot{X}_1, \ldots, \dot{X}_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r, t) - \\ - L(x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_r t) - \\ - \sum_s (\dot{X}_s - \dot{x}_s) \frac{\partial L}{\partial \dot{x}_s} \end{aligned} \tag{5.2}$$

Here x_s and u_k are functions which make the functional J a minimum, and X_s and U_k are any admissible functions satisfying the equations of the problem.

The relation (5.2) is valid for any type of restrictions on the coordinates both in the interior and on the boundary of the region X^*. In calculations we must substitute into the expression for E the function L with the appropriate superscript minus (-), plus (+), or zero (0), for which we should use the formulas given in the preceding two sections. These superscripts are omitted in the equality (5.2), as are the limits of the sums in its right-hand member. It should also be remembered that the number of the multipliers λ_s and μ_k may be different in different cases.

Substituting the expression for L into (5.2) and making use of the inequality (5.1), we obtain the following form for Weierstrass's necessary condition:

$$\begin{aligned} H_\lambda(x_1, \ldots, x_n, U_1, \ldots, U_m, \lambda_1, \ldots, \lambda_n, t) \leqslant \\ \leqslant H_\lambda(x_1, \ldots, x_n, u_1, \ldots, u_m, \lambda_1, \ldots, \lambda_n, t) \end{aligned} \tag{5.3}$$

where the identity $\dot{H}_\mu \equiv 0$ and the equality $\psi_{r+1}{}^{0} = 0$ are taken into consideration.

The results obtained here for systems with restrictions of the first type on the coordinates may be formulated in a manner similar to that given in [1, 2], where the corresponding problem was solved by methods making use of the construction of Pontriagin's maximum principle.

The above equations and relations were written in a form somewhat more complicated than is necessary for the problems considered. This minor complication (introduction of the superscripts zero (0), minus (-) and plus (+)) makes it possible to study more complicated problems, such as the problem of optimizing control processes in systems with bounded coordinates described by equations with discontinuous right-hand sides. The above described results are valid in these cases without any significant alterations. In [5] they are used in the calculation of optimal modes of operation of vibrotransporters.

6. Synthesis of optimal systems. The general mathematical formulation of the problem of synthesis of optimal systems with bounded control parameters is described in [2,10]. It is stated as the problem of constructing functions $v_k = v_k(x_1, \ldots, x_n, t)$ which define the values of the control parameters at each point of the space $x_1, \ldots, x_n, t$ in such a way that a system motion described by Equations (2.1) and (2.2) makes the functional J a minimum for $u_k = v_k$, for any initial values of the coordinates.

This statement of the problem, taking into account the restrictions on the coordinates and Equations (2.3), may be extended without any significant changes to the problems studied here. In the present section we note some properties of the problem of synthesizing optimal systems with bounded coordinates and control parameters, which distinguish it from the similar problem involving restrictions only on the control parameters. For this reason we shall distinguish the problem of synthesizing optimal control parameters for a system with restrictions on the coordinates and on the control parameters from the similar problems obtained from it by a transition to an open region of coordinate variation. The results of solving the latter, if a solution exists and can be constructed, remain valid for any optimal trajectories lying entirely in the interior of the region X^* and even for trajectories which touch the boundary of this region at a finite number of points.

For trajectories containing segments which lie entirely on the boundary of the region X^*, the problem of synthesis becomes considerably more complicated. Thus, for example, if there are restrictions of the first type on the coordinates and if the system moves along the boundary, then the equations describing the behavior of the system within the region X^* must be supplemented by equation of the form (1.5); the functions u_k may not satisfy this equation. In this case, therefore, the synthesizing functions must be constructed separately for the segments of the optimal trajectories which lie within the region X^* and for the segments which lie on its boundary.

The same process must also be repeated for the general formulation of optimization problems if there are restrictions of the second type on the coordinates. In this case, however, there evidently exists a sufficiently broad class of systems for whose synthesis a solution is provided by the solution of the corresponding problem with no restrictions on the coordinates.

This will be the case when to the segments of optimal trajectories lying in the interior of the region X^* we may assign trajectories which are optimal for the problem with no restrictions on the coordinates and which entirely contain the former trajectories and when the control parameters vary on the boundary of the region X^* in the same way as the open region of variation for the coordinates. Such an agreement between the two rules for variation of the control parameters may evidently be encountered in optimization problems with closed regions of variation of the control parameters in which the control parameters in which the control parameters in optimal modes of operation assume only boundary values.

7. **Example.** As an example to explain some of the general situations formulated above, we shall consider the problem of optimizing the length of time for transition from the state $\varphi(0) = \varphi^0$, $\xi(0) = \xi^0$ to the equilibrium condition $\varphi(T) = \xi(T) = 0$ for a simple system of indirect control of an astatic object, described by the equations

$$T_a\dot{\varphi} = \xi, \qquad T_s\dot{\xi} = u \qquad (|u| \leqslant 1) \tag{7.1}$$

in which φ is the input coordinate of the object, ξ is the coordinate of the control unit, $|u| \leqslant 1$ is the input value of the amplifier (control parameter).

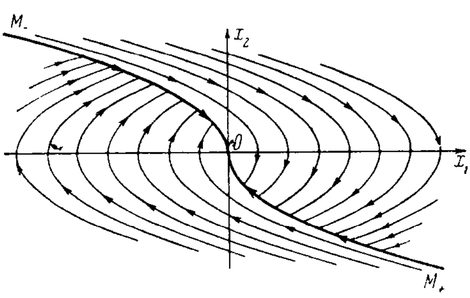

Fig. 1.

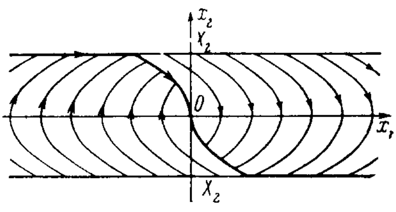

Fig. 2.

We introduce the notation $x_1 = T_aT_s\varphi$, $x_2 = T_s\xi$ and we formulate the optimization problem in the following manner: among the functions x_1, x_2, u and v, which satisfy the equations

$$g_1 = x_1 - x_2 = 0, \qquad g_2 = x_2 - u = 0, \qquad \psi = u^2 + v^2 - 1 = 0 \tag{7.2}$$

and the relations $\varphi_1 = x_1(T) = 0$, $\varphi_2 = x_2(T) = 0$, select those which make the functional $J = T$ a minimum.

Here, in order to avoid superfluous Lagrangean multipliers ρ_l, the left end of the trajectory is considered fixed [3] and we introduce the function ψ which realizes a transition to the open region of variation of u and the additional control parameter v. This problem, with no restrictions on the coordinates, has been fairly thoroughly studied [2, 11].

We construct the functions H and φ

$$H = H_\lambda + H_\mu = \lambda_1 x_2 + \lambda_2 u + \mu (u^2 + v^2 - 1) \tag{7.3}$$

$$\varphi = T + \rho_1 x_1 (T) + \rho_2 x_2 (T) \tag{7.4}$$

by means of which we construct the equations

$$\dot{\lambda}_1 = 0, \quad \dot{\lambda}_2 = -\lambda_1, \quad \lambda_2 + 2\mu u = 0, \quad 2\mu v = 0 \tag{7.5}$$

and the Erdmann-Weierstrass conditions [3]

$$\lambda_1^+ (t^*) = \lambda_1^- (t^*), \qquad \lambda_2^+ (t^*) = \lambda_2^- (t^*) \qquad (H_\lambda^-)_{t^*} = (H_\lambda^+)_{t^*} \tag{7.6}$$

valid in the open region of variation of the coordinates, as well as the equalities

$$\lambda_1 (T) = -\rho_1, \qquad \lambda_2 (T) = \rho_2 \tag{7.7}$$

The solution of the differential equations (7.5) which satisfies the conditions (7.7) is of the form

$$\lambda_1 = -\rho_1, \qquad \lambda_2 = \rho_1 (t - T) - \rho_2 \tag{7.8}$$

so that the function $\mu = -\lambda_2$ changes sign no more than once in the interval $t_0 \leqslant t \leqslant T$. Furthermore, we have $v = 0$ for $\mu \neq 0$. Consequently, $u = \pm 1$ everywhere except at the single point $t = t^*$, at which $\lambda_2(t^*) = 0$.

Beyond this the solution of the problem is found in the same way as, for example, in [2, 11]. As a result we obtain a family of optimal trajectories, shown in Fig. 1.

For example, let the coordinate x_2 be restricted by the inequality $|x_2| \leqslant x_2$. This restriction defines a strip of width $2X_2$, as shown in Fig. 2. The optimal trajectories located within this strip will consist of two parabolas. In such a system there may also be modes of operation in which x_2 assumes values on the boundary $x_2 = \pm X_2$; the last stage of the motion will follow the branches of the parabolas M_-0 and M_+0 passing through the origin.

For restrictions of the first type $u = 0$; for restrictions of the second type the control parameter can be arbitrary and may take on a value corresponding to the previous stage of the motion. Motion along the boundary in both cases will be described by the equation $\dot{x}_1 = \pm X_2$, where the lower sign corresponds to the boundary $x_2 = - X_2$ and the upper sign to the equality $x_2 = X_2$.

If the interval of permissible variation of the coordinate x_1 is defined by the inequality $|x_1| \leqslant X_1$, the problem is solved in a manner similar to the previous one. The corresponding constructions are shown in Fig. 3. They we made for restrictions of the second type. For restrictions of the first type the condition $\dot{x}_1 = 0$ $(x = \pm X_1)$ cannot be satisfied.

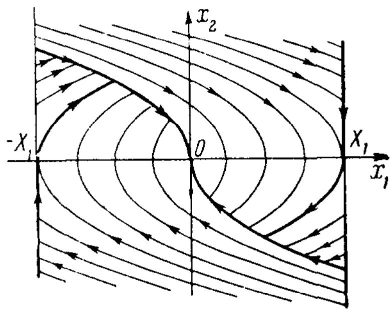

Fig. 3.

Motion along the boundary is described by the equation $\dot{x}_2 = u$, $u = \pm 1$; since for the boundary points

$$H^\circ = \lambda_2^\circ u^\circ + \mu^\circ(u^2 + u^2 - 1)$$

and consequently

$$\lambda_2^\circ = 0 \qquad \mu^\circ = -\frac{\lambda_2^\circ}{2u}, \qquad 2\mu^\circ u^\circ = 0$$

or

$$\lambda_2^\circ = \text{const}, \quad u = 0$$

This motion continues until the coordinate x_2 becomes zero, which is shown in Fig. 3. In accordance with the Erdmann-Weierstrass condition, $\lambda_2^\circ = \lambda_2^-(t_1)$. Consequently, by Weierstrass's condition the sign of the control parameter u will not change when x_1 passes to the boundary of the region $|x| \leqslant X_1$.

In both of the above cases the "switching" of the control parameter takes place along the curves M_-0 and M_+0, which define switching of the control parameters in a system without restrictions on the control parameters. Consequently, if there are restrictions of the second type, the synthesizing functions will be found from the solution of the problem having restrictions only on the control parameters.

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